

GENERALIZED SPLIT-OCTONION ELECTRODYNAMICS

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Abstract

Starting with the usual definitions of octonions and split octonions in terms of Zorn vector matrix realization, we have made an attempt to write the consistent form of generalized Maxwell's equations in presence of electric and magnetic charges (dyons). We have thus written the generalized potential, generalized field, and generalized current of dyons in terms of split octonions and accordingly the split octonion forms of generalized Dirac Maxwell's equations are obtained in compact and consistent manner. This theory reproduces the dynamic of electric (magnetic) in the absence of magnetic (electric) charges.

Key Words: Split octonions, Zorn Vector Matrix Realizations, Monopoles and dyons
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1 Introduction

The relationship between mathematics and physics has long been an area of interest and speculation. So, there has been a revival in the formulation of natural laws so that there exists [1] four-division algebras consisting the algebra of real numbers (\mathbb{R}), complex numbers (\mathbb{C}), quaternions (\mathbb{H}) and Octonions (\mathcal{O}). All four algebra's are alternative with totally anti symmetric associators. Quaternions [2, 3] were very first example of hyper complex numbers have been widely used [4, 5, 6, 7, 8, 9, 10] to the various applications of mathematics and physics. Since octonions [11] share with complex numbers and quaternions, many attractive mathematical properties, one might expect that they would be equally as useful as others. Octonion [11] analysis has been widely discussed by Baez [12]. It has also played an important role in the context of various physical problems [13, 14, 15, 16] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [17, 18, 19, 20] towards the developments of wave equation and octonion form of Maxwell's equations. We [21, 22] have also studied octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation. Keeping in view the recent interests on the existence of monopoles and dyons at one end and quaternion-octonion formulation of generalized Dirac-Maxwell's (GDM) equations at other end, we [23] have reformulated the generalized Dirac-Maxwell's equations of dyons by means of octonion variables in compact and consistent manner. Starting with the usual definitions of octonions and split octonions in terms of Zorn vector matrix realization, in this paper, we have made an attempt to extend the present study to write the consistent form of generalized Dirac-Maxwell's (GDM) equations of dyons in terms of split octonion variables without imposing constraints on physical variables as we [23] did for the case of normal octonions. We have thus written the generalized potential, generalized field, and generalized current of dyons in terms of split octonions and accordingly the split octonion forms of generalized Dirac Maxwell's equations are derived in compact and consistent manner. This theory reproduces the dynamic of electric (magnetic) in the absence of magnetic (electric) charges.

2 Octonion Definition

An octonion x is expressed [21, 22] as a set of eight real numbers

$$x = (x_0, x_1, \dots, x_7) = x_0 e_0 + \sum_{A=1}^7 x_A e_A \quad (A = 1, 2, \dots, 7) \quad (1)$$

where $e_A (A = 1, 2, \dots, 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0 = 1, \quad e_0 e_A = e_A e_0 = e_A \quad e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C. \quad (A, B, C = 1, 2, \dots, 7) \quad (2)$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 i.e.

$$f_{ABC} = +1 = (123), (471), (257), (165), (624), (543), (736).$$

Here the octonion algebra \mathcal{O} is described over the algebra of rational numbers having the vector space of dimension 8. Octonion algebra is non associative. Octonion conjugate is thus defined as,

$$\bar{x} = x_0 e_0 - \sum_{A=1}^7 x_A e_A \quad (A = 1, 2, \dots, 7). \quad (3)$$

The norm of the octonion $N(x)$ is defined as

$$N(x) = \bar{x}x = x\bar{x} = \sum_{\alpha=0}^7 x_\alpha^2 e_0 \quad (4)$$

which is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \quad (5)$$

As such, for a nonzero octonion x , we may define its inverse as $x^{-1} = \frac{\bar{x}}{N(x)}$ which shows that $x^{-1}x = xx^{-1} = 1.e_0$; $(xy)^{-1} = y^{-1}x^{-1}$.

3 Split Octonions

The split octonions are a non assosiative extension of split quaternions. .They differ from the octonion in the signature of quadratic form. the split octonions have a signature (4, 4) whereas the octonions have positive signature (8, 0). The Cayley algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{C}$ is visualized as the algebra of split octonions with its following basis elements,

$$\begin{aligned}
u_0 &= \frac{1}{2} (1 + i e_7), & u_0^* &= \frac{1}{2} (1 - i e_7), \\
u_1 &= \frac{1}{2} (e_1 + i e_4), & u_1^* &= \frac{1}{2} (e_1 - i e_4), \\
u_2 &= \frac{1}{2} (e_2 + i e_5), & u_2^* &= \frac{1}{2} (e_2 - i e_5), \\
u_3 &= \frac{1}{2} (e_3 + i e_6), & u_3^* &= \frac{1}{2} (e_3 - i e_6),
\end{aligned} \tag{6}$$

where (\star) is used for complex conjugation for which $(i = \sqrt{-1})$ is usual complex imaginary number and commutes with all the seven octonion imaginary units $e_A (A = 1, 2, \dots, 7)$. The split octonion basis elements satisfy the following multiplication rules [13];

$$\begin{aligned}
u_i u_j &= \epsilon_{ijk} u_k^*; & u_i^* u_j^* &= -\epsilon_{ijk} u_k (\forall i, j, k = 1, 2, 3) \\
u_i u_j^* &= -\delta_{ij} u_0; & u_i u_0 &= 0; & u_i^* u_0 &= u_i^* \\
u_i^* u_j &= -\delta_{ij} u_0; & u_i u_0^* &= u_0; & u_i^* u_0^* &= 0 \\
u_0 u_i &= u_i; & u_0^* u_i &= 0; & u_0 u_i^* &= 0 \\
u_0^* u_i^* &= u_i; & u_0^2 &= u_0; & u_0^{*2} &= u_0^*; & u_0 u_0^* &= u_0^* u_0 = 0
\end{aligned} \tag{7}$$

as the bi-valued representations of quaternion units e_0, e_1, e_2, e_3 . We may thus introduce a convenient realization for the basis elements (u_0, u_i, u_0^*, u_i^*) in terms of Pauli spin matrices as

$$\begin{aligned}
u_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; & u_0^* &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\
u_i &= \begin{bmatrix} 0 & 0 \\ e_j & 0 \end{bmatrix}; & u_j &= \begin{bmatrix} 0 & -e_j \\ 0 & 0 \end{bmatrix} (\forall j = 1, 2, 3)
\end{aligned} \tag{8}$$

where $1, e_1, e_2, e_3$ are quaternion units satisfying the multiplication rule $e_j e_k = -\delta_{jk} + \epsilon_{jkl} e_l$. The split Cayley octonion algebra is thus expressed in terms of 2×2 Zorn's vector matrix realizations as

$$A = a u_0^* + b u_0 + x_j u_j^* + y_j u_j = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix} \tag{9}$$

The split octonion conjugation of equation (9) is then described as,

$$\overline{A} = au_0 + bu_0^* - x_j u_j^* - y_j u_j = \begin{pmatrix} b & \vec{x} \\ -\vec{y} & a \end{pmatrix}. \quad (10)$$

The norm of A is defined as $\overline{A}A = A\overline{A} = (ab + \vec{x} \cdot \vec{y})\hat{1}$ with $\hat{1}$ as the unit matrix of order 2×2 . Any four - vector A_μ (complex or real) can equivalently be written in terms of the following Zorn matrix realization as

$$Z(A) = \begin{pmatrix} x_4 & -\vec{x} \\ \vec{y} & y_4 \end{pmatrix}; \quad Z(\overline{A}) = \begin{pmatrix} x_4 & \vec{x} \\ -\vec{y} & y_4 \end{pmatrix}. \quad (11)$$

So we may write the split octonion differential operator \square and its conjugate $\overline{\square}$ in terms of the 2×2 Zorn matrix realization as

$$\square = \begin{pmatrix} \partial_t & -\vec{\nabla} \\ \vec{\nabla} & -\partial_t \end{pmatrix}; \quad \overline{\square} = \begin{pmatrix} -\partial_t & \vec{\nabla} \\ -\vec{\nabla} & \partial_t \end{pmatrix}, \quad (12)$$

where $\partial_t = \frac{\partial}{\partial t}$. As such , we get

$$\square \overline{\square} = \begin{pmatrix} \nabla^2 - \frac{\partial^2}{\partial t^2} & 0 \\ 0 & \nabla^2 - \frac{\partial^2}{\partial t^2} \end{pmatrix} = \overline{\square} \square = \square, \quad (13)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}$.

4 Generalized Split Octonion Electrodynamics

Let us start with octonion form of generalized potential [22] of dyons as

$$\mathbb{V} = e_0 V_0 + e_1 V_1 + e_2 V_2 + e_3 V_3 + e_4 V_4 + e_5 V_5 + e_6 V_6 + e_7 V_7. \quad (14)$$

Here we have described the components of generalized four potential of dyons as

$$(V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7) \implies (\varphi, A_x, A_y, A_z, iB_x, iB_y, iB_z, i\phi) \quad (i = \sqrt{-1})$$

with $(\phi, A_x, A_y, A_z) = (\phi, \vec{A}) = \{A_\mu\}$ and $(\varphi, B_x, B_y, B_z) = (\varphi, \vec{B}) = \{B_\mu\}$ are respectively described as the components of electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four potential

constituents of dyons (particles carrying simultaneously the electric and magnetic charges). Equation (14) may then be written [23] as

$$\begin{aligned}\mathbb{V} &= e_1(A_x + ie_7 B_x) + e_2(A_y + ie_7 B_y) + e_3(A_z + ie_7 B_z) + (\varphi + ie_7 \phi) \\ &= e_1 V_x + e_2 V_y + e_3 V_z + ie_7 \emptyset.\end{aligned}\quad (15)$$

So, the split octonion form of generalized four potential of dyons may be written in terms of 2×2 Zorn's vector matrix realization as

$$\mathbb{V} = \begin{pmatrix} \Phi_- & -\vec{V}_+ \\ \vec{V}_- & \Phi_+ \end{pmatrix} = \begin{pmatrix} (\varphi - \phi) & -(\vec{A} + \vec{B}) \\ (\vec{A} - \vec{B}) & (\varphi + \phi) \end{pmatrix}. \quad (16)$$

Here $\left[\Phi_- = (\varphi - \phi), \Phi_+ = (\varphi + \phi), \vec{V}_- \rightarrow (\vec{A} - \vec{B}), \vec{V}_+ \rightarrow (\vec{A} + \vec{B}) \right]$ with $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$. Now operating $\overline{\square}$ (12) to octonion potential \mathbb{V} (16), we get

$$\overline{\square}\mathbb{V} = \begin{pmatrix} -\frac{\partial\varphi}{\partial t} + \frac{\partial\phi}{\partial t} + \vec{\nabla} \cdot \vec{A} - \vec{\nabla} \cdot \vec{B} & \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\varphi + \vec{\nabla}\phi - \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B} \\ -\vec{\nabla}\varphi + \vec{\nabla}\phi + \frac{\partial\vec{A}}{\partial t} - \frac{\partial\vec{B}}{\partial t} + \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B} & \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B} + \frac{\partial\varphi}{\partial t} + \frac{\partial\phi}{\partial t} \end{pmatrix}. \quad (17)$$

It is to be noted that we have used S.I. system of natural units ($c = \hbar = 1$) through out the text. Equation (14) then reduces to

$$\overline{\square}\mathbb{V} = \mathbb{F}; \quad (18)$$

where \mathbb{F} is also an octonion describing the generalized electromagnetic fields of dyons [23] as

$$\begin{aligned}\mathbb{F} &= \sum_{a=0}^{a=7} e_a F_a = e_1 F_1 + e_2 F_2 + e_3 F_3 + e_4 F_4 + e_5 F_5 + e_6 F_6 \\ &= e_1(H_x + ie_7 E_x) + e_2(H_y + ie_7 E_y) + e_3(H_z + ie_7 E_z)\end{aligned}\quad (19)$$

with the componets $F_0 = F_7 = 0$ due to Lorentz gauge conditions applied for electric and magnetic four potentials. Equation (19) may then be described as a split octonion written in terms of 2×2 Zorn's vector matrix realization as

$$\mathbb{F} = \begin{pmatrix} 0 & -\vec{F}_+ \\ \vec{F}_- & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(\vec{F}_g + \vec{F}_e) \\ (\vec{F}_g - \vec{F}_e) & 0 \end{pmatrix} \quad (20)$$

where $\vec{F}_+ = \vec{F}_g + \vec{F}_e$, $\vec{F}_- = \vec{F}_g - \vec{F}_e$ and

$$\begin{aligned} \vec{F}_g &= -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \phi + \vec{\nabla} \times \vec{A} \longrightarrow \vec{H}; \\ \vec{F}_e &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}_n \phi - \vec{\nabla}_n \times \vec{B}_n \longrightarrow \vec{E}. \end{aligned} \quad (21)$$

Here \vec{E} and \vec{H} are respectively described as the generalized electric and magnetic fields of dyons [22]. As such, we may write the generalized electromagnetic field vector \mathbb{F} of dyons in terms of following split octonionic representation i.e.

$$\mathbb{F} = \begin{pmatrix} 0 & -(\vec{H} + \vec{E}) \\ \vec{H} - \vec{E} & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\vec{\psi}_+ \\ \vec{\psi}_- & 0 \end{pmatrix} \quad (22)$$

where $\vec{\psi}_+ = \vec{H} + \vec{E}$ and $\vec{\psi}_- = \vec{H} - \vec{E}$ are described the generalized electromagnetic vector fields of dyons. Now operating the differential operator \square (12) to generalized vector field \mathbb{F} (22), we get

$$\square \mathbb{F} = \begin{pmatrix} \vec{\nabla} \cdot \vec{F}_g - \vec{\nabla} \cdot \vec{F}_e & \frac{\partial \vec{F}_g}{\partial t} + \frac{\partial \vec{F}_e}{\partial t} - \vec{\nabla} \times \vec{F}_g + \vec{\nabla} \times \vec{F}_e \\ \frac{\partial \vec{F}_g}{\partial t} - \frac{\partial \vec{F}_e}{\partial t} + \vec{\nabla} \times \vec{F}_g + \vec{\nabla} \times \vec{F}_e & \vec{\nabla} \cdot \vec{F}_g + \vec{\nabla} \cdot \vec{F}_e \end{pmatrix} \quad (23)$$

which may also be written in terms of the following wave equation in split octonion form as .

$$\square \mathbb{F} = -\mathbb{J}. \quad (24)$$

Here \mathbb{J} is also an octonion which may be identified as the octonion form of generalized four-current of dyons as

$$\mathbb{J} = \sum_{a=0}^{a=7} e_a \mathbf{j}_a = e_1(\mathbf{j}_x + ie_7 \mathbf{k}_x) + e_2(\mathbf{j}_y + ie_7 \mathbf{k}_y) + e_3(\mathbf{j}_z + \mathbf{k}_z) + (\varrho + ie_7 \rho). \quad (25)$$

It may also be written as the split octonion representation in terms of 2×2 Zorn's vector matrix realization as

$$\mathbb{J} = \begin{pmatrix} (\varrho - \rho) & -(\vec{j} + \vec{k}) \\ (\vec{j} - \vec{k}) & (\varrho + \rho) \end{pmatrix} = \begin{pmatrix} j_- & -\vec{j}_+ \\ \vec{j}_- & j_+ \end{pmatrix} \quad (26)$$

where $j_- = (\varrho - \rho)$, $\vec{j}_- = (\vec{j} - \vec{k})$, $j_+ = (\varrho + \rho)$, $\vec{j}_+ = (\vec{j} + \vec{k})$. Here $(\rho, \vec{j}) = \{\mathbf{j}_\mu\}$, $(\varrho, \vec{j}) = \{\mathbf{k}_\mu\}$ and $(J_0, \vec{J}) = \{\mathbf{J}_\mu\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons. Equations (24) thus leads to the generalized Dirac Maxwell's (GDM) equations of dyons [22]. Using equations (12,13 and 26), we get

$$\square \cdot \square \mathbb{V} = \begin{pmatrix} (\square \varphi - \square \phi) & -(\square \vec{A} + \square \vec{B}) \\ (\square \vec{A} - \square \vec{B}) & (\square \varphi + \square \phi) \end{pmatrix} = \begin{pmatrix} (\varrho - \rho) & -(\vec{j} + \vec{k}) \\ (\vec{j} - \vec{k}) & (\varrho + \rho) \end{pmatrix} = \mathbb{J} \quad (27)$$

which is described as the split potential wave equation for generalized fields of dyons. Now operating $\square \cdot$ (12) to split-octonion current \mathbb{J} (26), we get

$$\square \cdot \mathbb{J} = \begin{pmatrix} -\frac{\partial \varrho}{\partial t} + \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} - \vec{\nabla} \cdot \vec{k} & \frac{\partial \vec{j}}{\partial t} + \frac{\partial \vec{k}}{\partial t} + \vec{\nabla} \varrho + \vec{\nabla} \rho - \vec{\nabla} \times \vec{j} + \vec{\nabla} \times \vec{k} \\ -\vec{\nabla} \varrho + \vec{\nabla} \rho + \frac{\partial \vec{j}}{\partial t} - \frac{\partial \vec{k}}{\partial t} + \vec{\nabla} \times \vec{j} + \vec{\nabla} \times \vec{k} & \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \vec{k} + \frac{\partial \varrho}{\partial t} + \frac{\partial \rho}{\partial t} \end{pmatrix} \quad (28)$$

which can be written as

$$\square \cdot \mathbb{J} = \mathbb{S} \quad (29)$$

where

$$\mathbb{S} = \begin{pmatrix} (\mathfrak{S}_m - \mathfrak{S}_e) & -(\vec{r} + \vec{s}) \\ (\vec{r} - \vec{s}) & (\mathfrak{S}_m + \mathfrak{S}_e) \end{pmatrix} \mapsto \begin{pmatrix} \mathfrak{S}_- & \vec{S}_+ \\ \vec{S}_- & \mathfrak{S}_+ \end{pmatrix} \quad (30)$$

with $\mathfrak{S}_- = (\mathfrak{S}_m - \mathfrak{S}_e)$, $\vec{S}_- = (\vec{r} - \vec{s})$, $\mathfrak{S}_+ = (\mathfrak{S}_m + \mathfrak{S}_e)$, $\vec{S}_+ = (\vec{r} + \vec{s})$ and

$$\mathfrak{S}_m = \vec{\nabla} \cdot \vec{k} + \frac{\partial \varrho}{\partial t}; \quad \mathfrak{S}_e = \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} \quad (31)$$

along with

$$\vec{r} = -\vec{\nabla} \rho - \frac{\partial \vec{j}}{\partial t} - \vec{\nabla} \times \vec{k}; \quad \vec{s} = -\vec{\nabla} \varrho - \frac{\partial \vec{k}}{\partial t} + \vec{\nabla} \times \vec{j} \quad (32)$$

which are the split octonion current wave equations for the components of generalized fields of dyons. In equation (30) \mathfrak{S}_m and \mathfrak{S}_e are vanishing due to Lorentz gauge conditions applied for the cases of electric and magnetic charges.

As such, we have obtained consistently the generalized Dirac Maxwell's (GDM) equations from the theory of split octonion variables without constraints. The advantages of present formalism are discussed in terms of compact and simpler notations of split octonion valued generalized potential, generalized field and generalized currents of dyons. The present split octonion reformulation of generalized fields of dyons represents well the invariance of field equations under Lorentz and duality transformations. It also reproduces the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

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